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CONVECTIVE HEAT TRANSFER IN A BUBBLE LAMINAR FLOW IN A ROUND-CYLINDER TUBE

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The problem of heat exchange between a two-phase laminar flow in a round-cylinder tube and the isothermal wall of the tube is studied. An exact analytical solution is obtained.

We consider the problem of convective heat transfer in a thermal initial section in a two-phase laminar flow of liquid with monodisperse gas bubbles in a round-cylinder tube.

When $x < 0$, the tube walls are adiabatically insulated, whereas when $x \geq 0$ the wall temperature is kept constant, i.e., T_w = const, and is not equal to the liquid temperature T_{liq}, thus causing heat transfer which forms a nonuniform temperature field. It is obvious that at infinity the liquid temperature becomes equal to the wall temperature and heat transfer stops, $T_{liq}(r, \infty) = T_w$ = const. A laminar liquid flow in a tube of radius R is assumed to be dynamically stabilized; the temperatures of the phases are the same and the thermophysical characteristics of the media are constant.

The problem is described by the stationary equation of energy for each phase (subscripts 1 and 2 denote the carrying phase and the disperse phase, respectively):

$$(\rho_1 c_1 \mathbf{v}_1 \cdot \nabla)\,\theta = \lambda_1 \nabla^2 \theta\,, \quad (\rho_2 c_2 \mathbf{v}_2 \cdot \nabla)\,\theta = \lambda_2 \nabla^2 \theta\,, \tag{1}$$

where $\rho_i = \alpha_i \rho_i^0$, α_i is the volumetric content of the ith phase (in what follows $\alpha_2 = \alpha$), and $\theta(r, x) = T_w - T_{liq}(r, x)$.

We assume that the flow is hydrodynamically stabilized and the velocity distribution in the phases u_i is described by the Poiseuille profile:

$$u_i\,(r) = U_i \left(1 - \frac{r^2}{R^2}\right). \tag{2}$$

We assume that the density of the convective heat flux along the flow $q_{x\text{conv}} = \rho_i c_i u_i(r) \dfrac{\partial \theta}{\partial x}$ is much larger than the heat-flux density due to molecular heat conduction $q_{x\mu} = -\lambda_i \dfrac{\partial^2 \theta}{\partial x^2}$, i.e., $q_{x\text{conv}} >> q_{x\mu}$ (which holds at large Peclet numbers). Then, summing up Eqs. (1) and passing to the cylindrical coordinate system, we obtain

$$\frac{\partial \theta}{\partial x} = \frac{\lambda_{\text{eff}}}{\rho_1^0 (1 - \alpha)\, c_1 u_1\,(r) \left(1 + \dfrac{\rho_2^0}{\rho_1^0} \dfrac{\alpha}{1 - \alpha} \dfrac{c_2}{c_1} S\right)} \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r}\right), \tag{3}$$

where $\lambda_{\text{eff}} = \lambda_1 + \lambda_2$ is the coefficient of effective thermal conductivity of the bubble structure and $S = u_2/u_1$ (below, S = const). In the case of a bubble monodisperse flow, the effective thermal conductivity is calculated by the expression [1]

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$$\lambda_{\text{eff}} = \frac{\lambda_1^0}{1 + \frac{3}{2}\frac{\left(1 - \frac{\lambda_2^0}{\lambda_1^0}\right)\alpha}{1 - \left(1 - \frac{\lambda_2^0}{\lambda_1^0}\right)\sqrt[3]{\frac{9\pi}{16}\alpha^2}}} .$$

Hereinafter we take α = const.

Equation (3) is solved with boundary conditions of first kind:

$$x = 0 : \ \theta(r, 0) = F(r) , \quad r = 0 : \ \left.\frac{\partial\theta}{\partial r}\right|_{r=0} = 0 , \quad r = R : \ \theta(0, x) = 0 . \tag{4}$$

The first of them specifies the temperature distribution in the flow at the channel inlet (from this point on $F(r)$ is an even function), the second condition requires the unknown function to be symmetric relative to the tube axis, and the third condition denotes that on the tube wall the liquid temperature coincides with the wall temperature.

Problem (3), (4) is solved by the Fourier method [2]; for this purpose we represent the unknown function in the form of the product $\theta(r, x) = \Phi(x)\Psi(r)$. Then, the variables in Eq. (3) are separated and the equation takes the form

$$\frac{\text{Pe}}{R}\frac{\Phi'(x)}{\Phi(x)} = \frac{\Psi''(r) + \frac{1}{r}\Psi'(r)}{\left(1 - \left(\frac{r}{R}\right)^2\right)\Psi(r)} = -l^2 , \quad \text{Pe} = \frac{U_1 R}{\dfrac{\lambda_{\text{eff}}}{\rho_1^0 c_1 (1 - \alpha)\left(1 + \frac{\rho_2^0}{\rho_1^0}\frac{c_2}{c_1}\frac{\alpha}{1 - \alpha} S\right)}} . \tag{5}$$

The left-hand side of this equation depends solely on x and the right-hand side solely on r; therefore, the equality is possible when separately these parts are equal to the same constant, which we denote in terms of $-l^2$ (the separation constant must be negative, otherwise solutions which diverge at infinity are obtained).

Equation (5) disintegrates into two ordinary differential equations:

$$\Phi'(x) + \frac{l^2}{\text{Pe}} R\Phi(x) = 0 , \tag{6}$$

$$\Psi''(r) + \frac{1}{r}\Psi'(r) + l^2\left(1 - \left(\frac{r}{R}\right)^2\right)\Psi(r) = 0 . \tag{7}$$

The general solution of (6) has the form

$$\Phi(x) = C \exp\left(-\frac{l^2 R}{\text{Pe}} x\right), \tag{8}$$

where C is the constant to be determined.

To solve (7) we use a new independent variable $\eta = r/R$ and reduce the equation to the form of the Sturm–Liouville equation [2]:

$$\eta^2 \frac{d^2\Psi}{d\eta^2} + \eta\frac{d\Psi}{d\eta} + \eta^2\gamma(1 - \eta^2)\Psi = 0 , \tag{9}$$

where $\gamma = l^2R^2$.

We seek the solution of (9) in the form of the sum of the generalized power series

$$\Psi(\eta) = \sum_{p=0}^{\infty} A_p \eta^{n+p} . \tag{10}$$

Differentiating this series twice and substituting the corresponding derivatives into Eq. (9), we obtain the following condition for determining the coefficients A_p of the power series:

$$\sum_{p=0}^{\infty} (n+p-1)(n+p) A_p \eta^{n+p} + \sum_{p=0}^{\infty} (n+p) A_p \eta^{n+p} + \gamma \sum_{p=0}^{\infty} A_p \eta^{n+p+2} - \gamma \sum_{p=0}^{\infty} A_p \eta^{n+p+4} = 0 .$$

Comparison of the coefficients at the same powers of η allows the construction of the system of recurrent equations

$$(n-1)\, nA_0 + nA_0 = 0 ,$$

$$n(n+1) A_1 + (n+1) A_1 = 0 ,$$

$$(n+1)(n+2) A_2 + (n+2) A_2 + \gamma A_0 = 0 ,$$

$$(n+2)(n+3) A_3 + (n+3) A_3 + \gamma A_1 = 0 ,$$

$$(n+3)(n+4) A_4 + (n+4) A_4 + \gamma A_2 - \gamma A_0 = 0 ,$$

....

$$(n+p)^2 A_p + \gamma A_{p-2} - \gamma A_{p-4} = 0 .$$

If $A_0 \neq 0$ (it can be taken equal to unity), then it follows from the first equation of the system that $n = 0$. Under this condition, we sequentially find from the system of equations that all coefficients A_{2p+1} with odd powers are zero, and for the coefficients A_{2p} with even powers we obtain the recurrent formula

$$A_{2p} = \frac{\gamma}{(2p)^2} (A_{2p-4} - A_{2p-2}) ,$$

from which we sequentially find the first several coefficients of the series (here we additionally take $A_{-2} = 0$):

$$A_0 = 1 , \quad A_2 = -\frac{\gamma}{2^2} , \quad A_4 = \frac{\gamma}{4^2}\left(1 + \frac{\gamma}{2^2}\right), \quad A_6 = -\frac{\gamma}{6^2}\left[\frac{\gamma}{2^2} + \frac{\gamma}{4^2}\left(1 + \frac{\gamma}{2^2}\right)\right], \ldots .$$

The power series with the coefficients A_{2p} is alternating; moreover, at $\lim\limits_{p\to\infty} (A_{2p}) = 0$ and $\lim\limits_{n\to\infty} \frac{A_{2p+2}}{A_{sp}} < 1$, according to the Leibniz theorem, it is converging.

At $n = 0$ and $A_0 = 1$ we obtain the partial solution of Eq. (9)

$$\Psi\left(\frac{r}{R}\right) = \sum_{p=0}^{\infty} A_{2p} \eta^{2p} = \sum_{p=0}^{\infty} A_{2p} \left(\frac{r}{R}\right)^{2p} = 1 + \sum_{p=1}^{\infty} A_{2p} \left(\frac{r}{R}\right)^{2p} ;$$

then the partial solution (3) can be written as

$$\theta\left(\frac{r}{R}, x\right) = C \exp\left(-\frac{\gamma}{\mathrm{Pe}}\frac{x}{R}\right)\left(1 + \sum_{p=1}^{\infty} A_{2p}\left(\frac{r}{R}\right)^{2p}\right). \tag{11}$$

To find the separation constant γ, we address ourselves to the conditions on the wall $r = R$, $\theta(1, x) = 0$, i.e.,

$$\theta\,(1, x) = C \exp\left(-\frac{\gamma}{\mathrm{Pe}}\frac{x}{R}\right)\left(1 + \sum_{p=1}^{k} A_{2p}\right) = 0\,,$$

whence it follows that the coefficients of the series must satisfy the condition

$$1 + A_2 + A_4 + \ldots + A_{2k} = 0\,. \tag{12}$$

The solution of (12) with respect to γ gives a set of eigenvalues of the separation constant γ_n. This equation can be solved rather simply at small natural k:

$$k = 1:\quad 1 + A_2 = 1 - \frac{\gamma}{2^2} = 0\,,\quad \gamma_0 = 4\,;$$

$$k = 2:\quad 1 + A_2 + A_4 = 1 - \frac{\gamma}{2^2} + \frac{\gamma}{4^2}\left(1 + \frac{\gamma}{2^2}\right) = 0\,,\quad \gamma_0 = \begin{pmatrix} 6 + i2\sqrt{7} \\ 6 - i2\sqrt{7} \end{pmatrix};$$

$$k = 3:\quad 1 + A_2 + A_4 + A_6 = 1 - \frac{\gamma}{2^2} + \frac{\gamma}{4^2}\left(1 + \frac{\gamma}{2^2}\right) - \frac{\gamma}{6^2}\left(\frac{\gamma}{2^2} + \frac{\gamma}{4^2}\left(1 + \frac{\gamma}{2^2}\right)\right) = 0\,;$$

$$\gamma_0 = \begin{pmatrix} \dfrac{4}{3}a + \dfrac{16}{3} - \dfrac{260}{3a} \\ -\dfrac{2}{3} + \dfrac{16}{3} + \dfrac{130}{3a} \pm i2\sqrt{3}\left(\dfrac{a}{3} + \dfrac{65}{3a}\right) \\ a = \sqrt[3]{64 + 9\sqrt{3441}} \end{pmatrix}.$$

However, at large k, even the use of numerical procedures for determining the roots of the equation indicated becomes technically difficult due to its cumbersome representation. Nevertheless, use of recursion allows one to overcome this difficulty by the program. We consider the algorithm for determining the roots of the characteristic equation.

First we describe the function $A(\gamma, p)$ of the coefficients of the series

$$A\,(\gamma, p) = \left|\begin{array}{l} x \leftarrow 0\,,\ \text{ if }\ \operatorname{mod}(p, 2) = 1\ \text{ or }\ p < 0\,, \\ x \leftarrow 1\,,\ \text{ if }\ p = 0\,, \\ x \leftarrow -\dfrac{\gamma}{4}\,,\ \text{ if }\ p = 2\,, \\ x \leftarrow \dfrac{\gamma}{p^2}\,[A\,(\gamma, p-4) - A\,(\gamma, p-2)]\ \text{ in all the remaining cases}\,, \\ x\,. \end{array}\right.$$

On the first line, a zero is assigned to the value of the function if the argument p is a negative or odd number; the second and third lines of the program determine the values of the basic coefficients A_0 and A_2; the fourth line of the program-function uses the recursion — the determination of the function in terms of itself; the last line provides deduction of the function value. We note that reference to this function with the indicated value of k in the regime of symbol calculations of the MathCad environment displays the corresponding coefficients.

Then we determine the function

$$F(\gamma, k) = \sum_{p=0}^{k} A(\gamma, 2p) = A(\gamma, 0) + A(\gamma, 2) + ... + A(\gamma, 2k)$$

and apply the procedure of the search for the root $(F(\gamma, k), \gamma)$ of the equations to it; as a result, we obtain all real roots of the equation.* In what follows, we give the results of the search for the first real root (12) by the presented algorithm for different exponents k:

k	1	3	5	10	15	20
γ_0	4	6.205	7.193	7.313	7.313	7.313

Hence it follows that starting from the tenth-power equation ($k = 10$), the first real root ceases to change; consequently $\gamma_0 = 7.313$ is one of the roots of the equation $F(\gamma, \infty) = 0$.

Equation (12) is the algebraic equation of power k relative to γ. The coefficients of this polynomial are real alternating numbers; therefore, according to the principal theorem of algebra this equation has just k roots; some of them are complex-conjugate. For example, at $k = 8$ all roots of the equation are given as a table:

$$\gamma_0 = \begin{pmatrix} -187.9 \mp i95.95 \\ -12.77 \mp i75.19 \\ 7.309 \\ 29.07 \mp i130.81 \\ 31.89 \end{pmatrix}.$$

Each real positive root of (11) γ_n, $n = 1, 2, ..., k$, determines the coefficients of the series A_{2p}, $p = 0, 1, ..., k$, and, consequently, generates a partial solution of the equation of energy:

$$\theta_n(r, x) = C_n \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}}\frac{x}{R}\right) \sum_{p=0}^{k} A_{2p}(\gamma_n)\left(\frac{r}{R}\right)^{2p}. \tag{13}$$

We look for the general solution in the form of the sum of partial solutions:

$$\theta(r, x) = \sum_{n=1}^{m} \theta_n(r, x) = \sum_{n=1}^{m} C_n \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}}\frac{x}{R}\right) \sum_{p=0}^{k} A_{2p}(\gamma_n)\left(\frac{r}{R}\right)^{2p}, \tag{14}$$

where m is the number of real positive roots of Eq. (12), $k \to \infty$, and $m \le k$. To find the coefficients C_n of this series, we refer to the conditions at the channel inlet: $x = 0$, $\theta(r, 0) = F(r)$. Taking in (14) $x = 0$, we obtain

$$\theta(r, 0) = \sum_{n=1}^{k} C_n \sum_{p=0}^{k} A_{2p}(\gamma_n)\left(\frac{r}{R}\right)^{2p}. \tag{15}$$

Let the initial distribution in the flow be an even function of the radius $\theta(r, 0) = F(r)$, $F(-r) = F(r)$; then the function $F(r)$ can be expanded into the Taylor series in terms of even powers of the radius:

$$F\left(\frac{r}{R}\right) = F(0) + \sum_{n=1}^{k} \frac{F^{(2n)}(0)}{(2n)!}\left(\frac{r}{R}\right)^{2n} + \frac{F^{(2k+2)}(\xi)}{(2k+2)!}(\xi)^{2k+2}, \quad 0 < \xi < 1.$$

* We used the operators of the software environment MathCad 7 Professional.

If $F(0) = \Theta$ is the excess temperature on the flow axis at the channel inlet, then, substituting in the left-hand side of (15) the expansion of the function $F(r/R)$, we write

$$\Theta + \sum_{p=1}^{k} \frac{F^{(2p)}(0)}{(2p)\,!} \left(\frac{r}{R}\right)^{2p} = \sum_{n=1}^{m} C_n \sum_{p=0}^{k} A_{2p}(\gamma_n) \left(\frac{r}{R}\right)^{2p} = \sum_{p=0}^{k} \left(\frac{r}{R}\right)^{2p} \sum_{n=1}^{m} C_n A_{2p}(\gamma_n) \,.$$

Comparing the coefficients at the same powers r/R, we obtain the following system of linear equations for determining the coefficients C_n:

$$C_1 + C_2 + \ldots + C_m = \Theta \,,$$

$$A_2(\gamma_1)\, C_1 + A_2(\gamma_2)\, C_2 + A_2(\gamma_3)\, C_3 + \ldots + A_2(\gamma_m)\, C_m = \frac{F^{(2)}(0)}{2\,!} \,,$$

$$A_4(\gamma_1)\, C_1 + A_4(\gamma_2)\, C_2 + A_4(\gamma_3)\, C_3 + \ldots + A_4(\gamma_m)\, C_m = \frac{F^{(4)}(0)}{4\,!} \,,$$

....

$$A_{2k-2}(\gamma_1)\, C_1 + A_{2k-2}(\gamma_2)\, C_2 + A_{2k-2}(\gamma_3)\, C_3 + \ldots + A_{2k-2}(\gamma_m)\, C_m = \frac{F^{(2k)}(0)}{(2k)\,!} \,.$$

Let $a_{ij} = A_{2i-2}(\gamma_i)$ be the elements of the square matrix $\mathbf{A} = \|a_{ij}\|$, C_j be the elements of the vector $\mathbf{C} = (C_j)$, and B_i be the elements of the vector-column $\mathbf{B}$ of the right-hand side ($i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$, $m \le k$); then the system of linear equations can be written in the matrix form $\mathbf{A}\cdot\mathbf{C} = \mathbf{B}$, and its solution as $\mathbf{C}^* = \mathbf{A}^{-1}\mathbf{B}$, where $\mathbf{A}^{-1}$ is the inverse matrix. The solutions of the system of equations are the sought-for coefficients of the series $\left\{C_1^*, C_2^*, \ldots, C_m^*\right\}$. Now, the temperature distribution in the flow is described by the function

$$\theta\left(\frac{r}{R}, x\right) = \sum_{n=1}^{m} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}}\,\frac{x}{R}\right) \sum_{p=0}^{k} A_{2p}(\gamma_n) \left(\frac{r}{R}\right)^{2p} , \tag{16}$$

with $k \to \infty$.

The temperature field found allows calculation of the heat-transfer coefficient on the tube wall: according to the Fourier law of heat conduction, the heat flux density on the wall is

$$q_{\mathrm{w}} = -\frac{\lambda_{\mathrm{eff}}}{R} \left.\frac{\partial\theta\left(\frac{r}{R}, x\right)}{\partial\left(\frac{r}{R}\right)}\right|_{r=R} = -\lambda_{\mathrm{eff}}\,\frac{2}{R} \sum_{n=1}^{m} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}}\,\frac{x}{R}\right) \sum_{p=1}^{k} p A_{2p}(\gamma_n) \,.$$

We calculate the same heat flux by the Newton–Richman law; however, the result of the calculation will depend on the determining temperature, more precisely, on the way of its determination.

If we take the maximum temperature between the wall and the flow axis as determining, for convective heat flux on the wall we should write

$$q_{\mathrm{w}} = -\beta_{\max}\,\theta(0, x) = -\beta_{\max} \sum_{n=1}^{k} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}}\,\frac{x}{R}\right),$$

where $\beta_{\max}$ is the heat-transfer coefficient calculated by the maximum temperature head.

Equating the calculated densities of heat fluxes and introducing the Nusselt number, we find

$$\mathrm{Nu}_{\max}(x) = \frac{\beta_{\max} 2R}{\lambda_{\mathrm{eff}}} = 4 \frac{\sum_{n=1}^{m} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}} \frac{x}{R}\right) \sum_{p=1}^{k} pA_{2p}(\gamma_n)}{\sum_{n=1}^{k} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}} \frac{x}{R}\right)} .$$

If we take the mean temperature within the range $[0, R]$ as determining

$$\bar{\theta}(x) = \frac{1}{R}\int_0^R \theta(r, x)\, dr = \sum_{n=1}^{m} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}} \frac{x}{R}\right)\left(1 + \sum_{p=1}^{k} \frac{A_{2p}(\gamma_n)}{2p+1}\right),$$

the Newton–Richman law acquires the form $q_{\mathrm{w}} = \beta\bar{\theta}$, and the Nusselt number becomes equal to

$$\mathrm{Nu}_{\mathrm{mid}}(x) = \frac{\beta 2R}{\lambda_{\mathrm{eff}}} = 4 \frac{\sum_{n=1}^{m} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}} \frac{x}{R}\right) \sum_{p=1}^{k} pA_{2p}(\gamma_n)}{\sum_{n=1}^{m} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}} \frac{x}{R}\right)\left(1 + \sum_{p=1}^{k} \frac{A_{2p}(\gamma_n)}{2p+1}\right)} .$$

Then, if we take the section-mean temperature as determining

$$\bar{\bar{\theta}}(x) = \frac{1}{S}\int_S \theta(r, x)\, dS = \frac{1}{\pi R^2}\int_0^R 2\pi r\theta(r, x)\, dr = \sum_{n=1}^{k} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}} \frac{x}{R}\right)\left(1 + \sum_{p=1}^{k} \frac{A_{2p}(\gamma_n)}{p+1}\right),$$

then

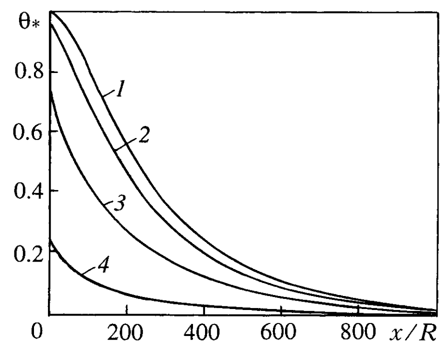
$$\mathrm{Nu}_{\min}(x) = \frac{\beta 2R}{\lambda_{\mathrm{eff}}} = 4 \frac{\sum_{n=1}^{k} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}} \frac{x}{R}\right) \sum_{p=1}^{k} pA_{2p}(\gamma_n)}{\sum_{n=1}^{k} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}} \frac{x}{R}\right)\left(1 + \sum_{p=1}^{k} \frac{A_{2p}(\gamma_n)}{p+1}\right)} .$$

Finally, if the mass-mean temperature is taken as determining

$$\langle\theta\rangle = \frac{1}{\dot{m}c_v}\int_S \rho u(r)\, c_v \theta(r, x)\, dS = \frac{4}{R^2}\int_0^R r\left(1 - \left(\frac{r}{R}\right)\right)^2 \theta(r, x)\, dr =$$

$$= \sum_{n=1}^{m} C_n^* \exp\left(-\frac{|\gamma_n|}{\mathrm{Pe}} \frac{x}{R}\right)\left(1 + 2\sum_{p=1}^{k} \frac{A_{2p}(\gamma_n)}{(p+1)(p+2)}\right),$$

then



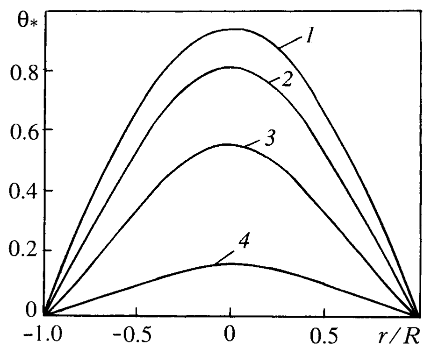


Fig. 1. Distribution of the reduced temperature θ_* on different radii along the channel length x/R: 1) $r/R = 0$; 2) 0.3; 3) 0.6; 4) 0.9.

Fig. 2. Distribution of the reduced temperature along the radius in different cross sections of the channel: 1) $x = 50R$; 2) $100R$; 3) $200R$; 4) $500R$.

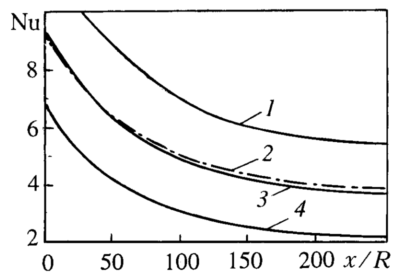


Fig. 3. Dependences of the Nusselt number on the channel length x/R: 1) $\mathrm{Nu_{min}}$; 2) Nu; 3) $\mathrm{Nu_{mid}}$; 4) $\mathrm{Nu_{max}}$.

$$\mathrm{Nu}\,(x) = \frac{\beta 2R}{\lambda_{\mathrm{eff}}} = 4\,\frac{\sum\limits_{n=1}^{m} C_n^* \exp\left(-\dfrac{|\gamma_n|}{\mathrm{Pe}}\dfrac{x}{R}\right)\sum\limits_{p=1}^{k} pA_{2p}\,(\gamma_n)}{\sum\limits_{n=1}^{m} C_n^* \exp\left(-\dfrac{|\gamma_n|}{\mathrm{Pe}}\dfrac{x}{R}\right)\left(1 + 2\sum\limits_{p=1}^{k}\dfrac{A_{2p}\,(\gamma_n)}{(p+1)\,(p+2)}\right)}\,.$$

Figure 1 shows the distribution of the reduced temperature $\theta(r/R,\ x)/\Theta$ along the channel for different radii. Calculations were made for water at $R = 0.005$ m, $\mathrm{Re}_1 = 2U_1R/\nu_1 = 1695$, and $\mathrm{Pe} = 1742$ with the initial temperature $\Theta = 100^{\circ}$C, and the basic system of functions was determined at $k = 8$. The real positive roots of (12) were $\gamma_1 = 7.3115$ and $\gamma_2 = 31.89$, and the solutions of the linear system of equations normalized by the maximum temperature were $C_1 = 1.2976$ and $C_2 = -0.2976$. In the calculation, the gas content was set as $\alpha_2 = 20\%$ and $S = 1$.

The character of variation of the excess temperature at any distance from the flow axis indicates its smooth variation along the flow; thus, thermal equilibrium is attained only at a distance of 500 gauges from the inlet section.

Figure 2 presents the graphs of the distribution of the reduced temperature along the radius for different sections x.

Figure 3 gives the dependences of the Nusselt number along the channel length x. The curves show that Nusselt numbers constantly decrease along the flow to the end of the section of thermal stabilization of the flow. The asymptotic values are: $\mathrm{Nu_{max}} = 2.133$, $\mathrm{Nu_{mid}} = 3.677$, $\mathrm{Nu_{min}} = 5.365$, and $\mathrm{Nu} = 3.816$. In the case of a one-phase flow with the same Reynolds number, calculation of the Nusselt number leads to $\mathrm{Nu} = 3.633$.

Comparison of the results obtained with those available in the literature can be made on the basis of the limiting value of gas content in the flow $\alpha_2 = 0\%$ at the end of the thermal section by the Nusselt number: calculation of this number for a one-phase liquid flow by the presented method gives $\mathrm{Nu}(\infty) = 3.633$; a value of $\mathrm{Nu}(\infty) = 3.658$ is given in [3]. This value was checked repeatedly in the experiments. In [3], this problem is solved by sequential verification of the temperature curve in the flow till it completely satisfies the balance equation of energy, which is not always convenient.

Calculation by the theory of a thermal boundary two-phase layer leads to close, but larger, values of Nu_{mid}; however, these results coincide with 10% accuracy.

NOTATION

α, volumetric content of the *i*th phase; *a*, thermal diffusivity, m^2/sec; *A* and *C*, coefficients of the series; **A**, matrix of the coefficients; **C**, row-vector; **B**, column-vector; C_m^*, solutions of the system of linear equations; β, heat-transfer coefficient, W/(m^2·K); *r*, *x*, independent variables of the cylindrical system of coordinates, m; *R*, tube radius, m; ρ, density, kg/m^3; *c*, specific mass heat capacity, J/(kg·K); c_i, specific mass isochoric heat capacity of the mixture, J/(kg·K); **v**, velocity vector, m/sec; θ, excess temperature, K; λ, thermal conductivity, W/(m·K); ν, kinematic coefficient of viscosity, m^2/sec; *u*, longitudinal velocity, m/sec; *T*, temperature, K; *U*, velocity on the flow axis, m/sec; Pe, Peclet number; Re, Reynolds number; Nu, Nusselt number; *q*, heat-flux density, W/m^2; *S*, coefficient of slip of phases; *F*(*r*), function specifying the temperature distribution at the tube inlet, K; $\Phi(x)$ and $\Psi(\eta)$, separation functions in the Fourier method; $-l^2$, separation constant; η, reduced coordinate; γ_n, roots of the characteristic equation; *m*, *k*, *n*, *p*, integer exponents; $\dot{m}$, mass flow rate of the mixture, kg/sec; *i*, imaginary unit. Indices: $i = 1$, liquid phase; $i = 2$, gas phase; w, surface; liq, liquid; eff, effective value; *x*, projection of the vector on the longitudinal direction; max, maximum; mid, mean; min, minimum; conv, convective; μ, molecular; 0, true value of the physical quantity in the heterogeneous mixture; primes, derivatives; asterisk, reduced quantities.

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